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Thermally-driven linear vortex

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We investigate steady axially symmetric small Rossby number flows in which the driving consists of prescribed axial heat sources. By letting the velocity be proportional to the shear at the bottom surface we study the effects of that boundary condition on the resulting flows.

A multi-boundary-layer structure is found in the core, surrounding the heat sources. That structure depends on the relative magnitudes of the aspect ratio, stratification parameter and Ekman number.

1. Introduction and formulation

We investigate the motion resulting from a distribution of heat sources prescribed along the axis of rotation of a body of fluid having small departures from a state of rigid rotation and stable stratification. We will call such a state the basic state. The fluid is bounded at $z' = 0$ by an infinite horizontal boundary and at $z' = h$ by an infinite free surface assumed to be horizontal. Such an assumption is justified if $\Omega^2 l/g \ll 1$ in which case all of the thermodynamic variables of the basic state depend only on z' and that flow can be taken to be steady.

We then propose to study a *well-posed linear* problem and understand the various flow régimes which are found for different values of the parameters entering the problem. Although one could seek a motivation for this work in the understanding of strong atmospheric vortices, our model does not claim to describe these natural phenomena but rather it develops our physical intuition as to the importance of the various mechanisms for the different flow régimes.

Write

$$q(z') = -\frac{1}{Q}\lim_{r'\to 0} r'\frac{\partial\theta'}{\partial r'} = -\frac{1}{Q}\lim_{r'\to 0} H', \tag{1.1}$$

where $(\ )'$ denotes dimensional quantities and $q(z')$ is the dimensionless $O(1)$ prescribed axial heat flux. Q has dimensions of temperature and H' is the heat flux. We take the Rossby number

$$\epsilon = \alpha g h Q/\Omega^2 l^2 \tag{1.2}$$

to be small compared to unity and use the linearized equations; α is the coefficient of volumetric expansion, g is gravity and l is some horizontal scale to be defined. The Ekman number

$$E = \nu_V/\Omega h^2 \tag{1.3}$$

measures the relative importance of the viscous to Coriolis forces. Here ν_V, ν_H are the vertical and horizontal eddy coefficients of viscosity. In natural phenomenon these coefficients depend upon the size of the turbulent eddies and upon the degree of stratification of the flow. Furthermore, in rotating flows some distributions of angular momentum are stable to lateral displacements and thus one would expect turbulence to be anisotropic. For simplicity we take ν_V and ν_H to be constant and we define a horizontal scale l, to be

$$l = h(\kappa_H/\kappa_V)^{\frac{1}{2}}, \tag{1.4}$$

where $\sigma = \nu_H/\kappa_H = \nu_V/\kappa_V$ is a single Prandtl number. This horizontal scale does not imply that the lateral boundary is at $r' = l$; rather we assume that all the physical quantities tend to zero as $r' \rightarrow \infty$.

We denote by λ the aspect ratio defined as

$$\lambda = h/l = (\kappa_V/\kappa_H)^{\frac{1}{2}} \tag{1.5}$$

and treat λ as a parameter.

Another parameter of importance is the stratification parameter S which is proportional to the ratio of the Brunt–Väisälä frequency squared

$$N^2 = \alpha g \Delta T/h$$

to the angular frequency of rotation squared Ω^2, i.e.

$$S = \sigma\lambda^2(\alpha g \Delta T/h\Omega^2) = \sigma\lambda^2 N^2/\Omega^2, \tag{1.6}$$

where ΔT is a measure of the temperature difference of the basic flow between the levels $z' = 0$ and $z' = h$. By allowing S to vary we will parametrize the importance of the effects of stratification relative to those due to rotation.

Finally, we formulate the boundary condition on the $z' = 0$ surface as follows: We allow the horizontal velocity vector to be proportional to the horizontal shear vector near $z' = 0$. By varying the constant of proportionality between the horizontal velocity vector and the shear vector we will be able to model flows having, at $z' = 0$, a rigid boundary or a free boundary as limiting cases. (For relevance of this type of boundary condition to atmospheric flows see Taylor (1915).)

For $E \ll 1$, boundary layers are found along the axis and along the horizontal boundaries. We shall use additive boundary-layer corrections which decay exponentially outside the boundary layer under consideration.

Then, define and scale quantities in the usual manner, i.e. write

$$\left.\begin{aligned} & r' = lr,\ z' = hz; \quad (u', v') = \epsilon\Omega l(u, v); \quad w' = \epsilon\Omega h w; \\ & \rho' = \rho'_S(z) - \alpha\rho_0 Q\theta; \quad T' = T_0 + \Delta T(z/h) + Q\theta, \\ & p' = p'_S(z) + \epsilon\Omega^2 l^2 \rho_0 p, \end{aligned}\right\} \tag{1.7}$$

where the subscript S denotes static quantities. The linearized Boussinesq equations read:

radial momentum,

$$2A = rp_r + ED^2\psi_z; \tag{1.8a}$$

angular momentum,

$$-2\psi_z = ED^2 A; \tag{1.8b}$$

vertical momentum,

$$0 = -p_z + E\lambda^2\nabla^2 r^{-1}\psi_r + \theta; \tag{1.8c}$$

heat equation,

$$r^{-1}\psi_r = ES^{-1}\nabla^2\theta; \tag{1.8d}$$

where

$$D_1^2 = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}; \quad D^2 \equiv D_1^2 + \frac{\partial^2}{\partial z^2}.$$

We have assumed the flow to be steady and axially symmetric and have introduced the stream function ψ such that

$$\psi_z = -ru, \quad \psi_r = rw, \tag{1.9}$$

and

$$A = rv, \quad H = r(\partial\theta/\partial r) \tag{1.10}$$

are the relative angular momentum and the heat flux.

For boundary conditions we demand that, at $r = 0$

$$A = \psi = 0, \quad \lim_{r\to 0} r^{-1}\psi_z = 0, \quad -\lim_{r\to 0} r\theta_r = q(z). \tag{1.11}$$

All quantities are to decay to zero as $r \to \infty$; (1.12)

$$\theta = 0, \quad \psi = 0, \quad \left(\frac{\partial}{\partial z} - \beta\right)(A, \psi_z) = 0 \quad \text{at} \quad z = 0; \tag{1.13}$$

and

$$\theta = 0, \quad A_z = \psi_{zz} = \psi = 0 \quad \text{at} \quad z = 1. \tag{1.14}$$

For $\beta = 0$, the $z = 0$ boundary is a free surface while for $\beta = \infty$ that boundary is a rigid surface.

We then propose to look at various flows obtained by varying λ, S as functions of the Ekman number E and by varying β. We will first consider in case 1 the simple problem in which $\beta = 0$; this corresponds to a model having two free surfaces. Case 2 then deals with the more general values of β, but with

$$E^{\frac{1}{2}} \ll S \ll E^{-\frac{1}{2}}.$$

Finally, in case 2 we consider a simple example to illustrate the dynamics.

2. Case 1, β = 0

Write θ, ψ, A as

$$\left.\begin{aligned} \theta(r,z) &= \sum_{n=1}^{\infty} \theta_n(r)\sin(\alpha_n z), \\ \psi(r,z) &= \sum_{n=1}^{\infty} \psi_n(r)\sin(\alpha_n z), \\ A(r,z) &= \sum_{n=1}^{\infty} A_n(r)\cos(\alpha_n z), \\ \text{and let} \quad q(z) &= \sum_{n=1}^{\infty} q_n\sin(\alpha_n z), \end{aligned}\right\} \tag{2.1}$$

where $\alpha_n = n\pi$. The above expressions satisfy all boundary conditions at $z = 0, 1$ (see (1.13), (1.14), $\beta = 0$). Thus there are no Ekman layers on the horizontal boundaries. In the expression for $A(r,z)$, we have omitted the mean angular momentum $\bar{A}$ where

$$\bar{A} = \int_0^1 dz\, A(r,z);$$

this term must be zero since no torques are applied at $r = 0, \infty$ and $z = 0, 1$. We will have more to say on this in case 2.

If we substitute (2.1) into the governing equations, we obtain a set of ordinary differential equations in r, namely,

$$2\alpha_n \psi_n + E(D_1^2 - \alpha_n^2) A_n = 0, \tag{2.2a}$$

$$S D_1^2 \psi_n - E(D_1^2 - \alpha_n^2) H_n = 0, \tag{2.2b}$$

$$2\alpha_n A_n + H_n + E(D_1^2 - \alpha_n^2)(\lambda^2 D_1^2 - \alpha_n^2)\psi_n = 0, \tag{2.2c}$$

where, for simplicity, H_n stands for the zonally averaged heat flux, i.e.

$$H_n = r\,\partial\theta_n/\partial r.$$

As boundary conditions we demand that, at $r = 0$,

$$\psi_n = A_n = D_1^2\psi_n = 0, \quad H_n + q_n = 0, \tag{2.3a}$$

and that

$$\psi_n, \quad A_n, \quad H_n, \quad D_1^2\psi_n \to 0 \quad \text{as} \quad r \to \infty. \tag{2.3b}$$

After we eliminate A_n, H_n we obtain a single equation for ψ_n, namely

$$E^2(D_1^2 - \alpha_n^2)^2(\lambda^2 D_1^2 - \alpha_n^2)\psi_n + S D_1^2 \psi_n - 4\alpha_n^2 \psi_n = 0, \tag{2.4}$$

with, at $r = 0$,

$$\psi_n = D_1^2 \psi_n = D_1^4 \psi_n - (q_n / E\lambda^2) = 0. \tag{2.5}$$

Now substitute

$$\psi_n(r) \propto (\mu r) K_1(\mu r)$$

into (2.4) and obtain a cubic equation for μ^2 which reads

$$E^2(\mu^2 - \alpha_n^2)^2(\mu^2\lambda^2 - \alpha_n^2) + S\mu^2 - 4\alpha_n^2 = 0. \tag{2.6}$$

It can be shown that (2.6) has no negative real roots μ^2.

Thus, by using the three roots of μ with $\operatorname{Re}\mu > 0$ we could construct an *exact* solution for $\psi_n(r)$ which would satisfy all of the boundary conditions at $r = 0, \infty$. $A_n(r)$ and $\theta_n(r)$ or $H_n(r)$ could then be determined since (2.2*a*) and (2.2*b*) are Poisson equations in which the meridional circulation redistributes the basic angular momentum and temperature stratification. The algebraic solution of (2.6) for the roots of that sextic tends to obscure the physics. Instead, we will assume $E \ll 1$ and use singular perturbation theory to find approximate expressions for the roots and therefore for ψ_n. Furthermore, we will only consider two terms to balance in (2.6) and we will discuss the six possible balances. The first term in (2.6), the viscous term, is much smaller than the last term except possibly when $|\mu| \gg 1$. A consistent lowest-order approximation is to neglect α_n^2 compared with μ^2 in the factor $(\mu^2 - \alpha_n^2)^2$. By taking $\alpha_n \sim O(1)$ we tacitly assume that all the energy is present in the lowest modes. This assumption is substantiated later on (see (3.18)). Then (2.6) reads

$$E^2\lambda^2\mu^6 - E^2\alpha_n^2\,\mu^4 + S\mu^2 - 4\alpha_n^2 = 0. \tag{2.7}$$

Each of these balances will hold in a given region of the parameter plane (λ, S). For two-terms of (2.7) to balance we must demand that the remaining ones are small compared to those retained. These restrictions will provide us with relations among λ, S and E which can be drawn as straight lines in the (λ, S) plane provided we measure quantities on both axes on a logarithmic scale. On these bounding lines more than two terms balance and the representation for the flow is more complex. These lines divide the first quadrant of the (λ, S) plane into four sectors as shown in figure 1. Table 1 summarizes the salient features of the dynamics of the core region. Column four indicates the number of degrees of freedom that each boundary layer satisfies. These degrees of freedom must always add up to three for any multi-boundary-layer structure of the axial region.

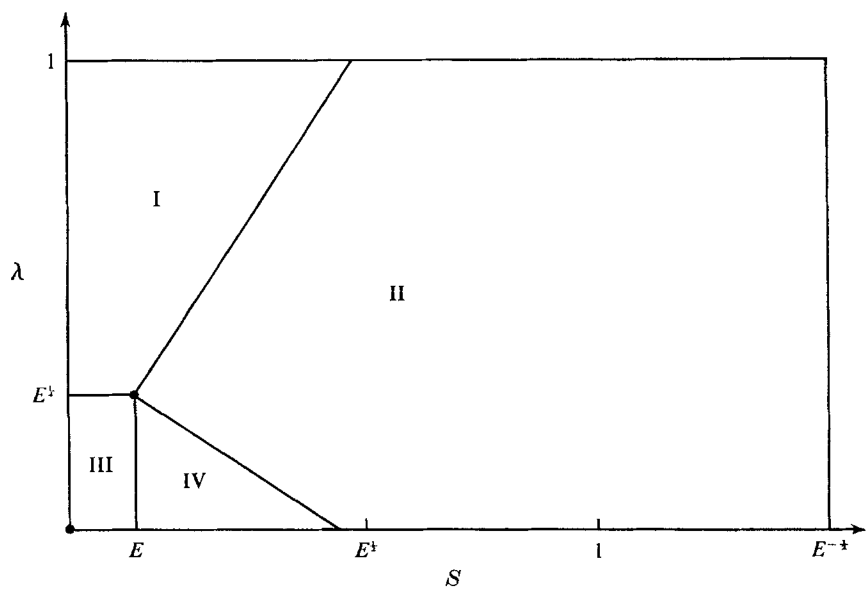

FIGURE 1. Boundary layers in parameter space.

Let us now consider the various possible two-term balances.

(i) $S\mu^2 - 4\alpha_n^2 = 0$, where

$$S \gg (E\lambda)^{\frac{2}{3}}, \quad S \gg E,$$

for the neglected terms in (3.7) to be small. These restrictions place us in sectors II and IV of figure 1. This is the hydrostatic layer (see Veronis 1967, and Barcilon & Pedlosky 1967*a*, *b*). Since there is only one root with $\mathrm{Re}\,\mu > 0$, this layer has only one degree of freedom which is used to satisfy the angular momentum boundary condition at $r = 0$.

(ii) $\alpha_n^2(E^2\mu^4 + 4) = 0$, with restrictions

$$S \ll E, \quad \lambda \ll E^{\frac{1}{2}},$$

which place us in sector III of figure 1. A layer of this type, called the upwelling layer, was considered by Pedlosky (1968) in the study of vertical transports of a homogeneous rotating fluid on the β plane. Two roots with $\mathrm{Re}\,\mu > 0$ enable us to accommodate two boundary conditions at $r = 0$.

Name	Dimensionless thickness	Physical balances	Restrictions and sector	Degrees of freedom	Relative orders of magnitude: $A = rv$	$H = r\theta_r$	ψ	rw_r
Hydrostatic	$(S)^{\frac{1}{2}}$	$-2v = -p_r$ $p_z = \theta$	$S \gg E$ $S \gg (E\lambda)^{\frac{2}{3}}$ Sectors II and IV	1	1	1	$\frac{E}{S}$	$\frac{E}{S^2}$
Stewartson	$(E\lambda)^{\frac{1}{3}}$	$-2v = -p_r$ $p_z = E\lambda^2\nabla_1^2 w$	$S \ll (E\lambda)^{\frac{2}{3}}$ $\lambda \gg E^{\frac{1}{2}}$ Sector I	3	1	$\frac{S}{(E\lambda)^{\frac{2}{3}}}$	$\frac{E}{(E\lambda)^{\frac{2}{3}}}$	$\frac{E}{(E\lambda)^{\frac{4}{3}}}$
Upwelling	$E^{\frac{1}{2}}$	$-2v = E(\nabla_1^2 - r^{-2})u$ $-2u = E(\nabla_1^2 - r^{-2})v$	$S \ll E$ $\lambda \ll E^{\frac{1}{2}}$ Sector III	2	1	$\frac{S}{E}$	1	$\frac{1}{E}$
Buoyancy	$\left(\frac{E^2\lambda^2}{S}\right)^{\frac{1}{4}}$	$0 = \theta + E\lambda^2\nabla_1^2 w$ $Sw = E\nabla_1^2\theta$	$S \gg (E\lambda)^{\frac{2}{3}}$ $S\lambda^2 \gg E^2$ Sector II	2	$\frac{E\lambda}{S^{\frac{3}{2}}}$	1	$\frac{E}{S}$	$\frac{1}{S\lambda^2}$
Viscous hydrostatic	$\frac{E}{S^{\frac{1}{2}}}$	$p_r = E\left(\nabla_1^2 - \frac{1}{r^2}\right)u$ $p_z = \theta$	$S \gg E$ $S\lambda^2 \ll E^2$ Sector IV	1	$\left(\frac{E}{S}\right)^2$	1	$\frac{E}{S}$	$\frac{1}{E}$
Stokes	λ	$p_r = E\left(\nabla_1^2 - \frac{1}{r^2}\right)u$ $p_z = E\lambda^2\nabla_1^2 w$	$\lambda \ll E^{\frac{1}{2}}$ $S\lambda^2 \ll E^2$ Sectors III and IV	1	$\left(\frac{\lambda}{E^{\frac{1}{2}}}\right)^4$	$\frac{S\lambda^2}{E^2}$	$\frac{\lambda^2}{E}$	$\frac{1}{E}$
Stewartson	$E^{\frac{1}{4}}$	$-2v = -p_r$ $p_z = 0$	$S \ll E^{\frac{1}{2}}$	—	1	$\frac{S}{E^{\frac{1}{2}}}$	$E^{\frac{1}{2}}$	$E^{\frac{1}{4}}$

TABLE 1. Boundary-layer summary

(iii) $(E^2\lambda^2\mu^6 - 4\alpha_n^2) = 0$, with restrictions

$$S \ll (E\lambda)^{\frac{2}{3}}, \quad \lambda \gg E^{\frac{1}{2}},$$

places us in sector I of figure 1. The layers arising in this region of parameter space were first discussed by Stewartson (1957) for homogeneous fluids; Barcilon & Pedlosky (1967*a*, *b*) showed their relevance in the presence of weak stratification. We now have three roots with $\mathrm{Re}\,\mu > 0$; thus this layer can account for the three axial boundary conditions.

All of the above balances contained the term $4\alpha_n^2$ which represents Coriolis forces. The remaining balances are then independent of the basic rotation.

(iv) $E^2\lambda^2\mu^4 + S = 0$, with restrictions

$$S \gg (E\lambda)^{\frac{2}{3}}, \quad S \gg (E/\lambda)^2,$$

puts us in sector II. This is the buoyancy layer in which viscous forces balance the buoyancy term and, in the energy equation, a balance is found between conduction of the perturbation field and advection of the basic temperature field. Such a layer was discussed by Barcilon & Pedlosky (1967*a*, *b*) and Veronis (1967). There are two roots for which $\mathrm{Re}\,\mu > 0$, i.e. this layer has two degrees of freedom.

(v) $\mu^2(-E^2\alpha_n^2\mu^2 + S) = 0$, with

$$(E/\lambda)^2 \gg S \gg E,$$

i.e. we are now in sector IV of figure 1. This layer and the next one, do not, to our knowledge, appear in the literature; we call these layers the Stokes layer and the viscous hydrostatic layer. In the viscous hydrostatic layer, viscous forces are dominant but the fluid is sufficiently stratified to allow for a hydrostatic balance in the vertical. Only one degree of freedom is associated with each layer.

(vi) $E^2\mu^4(\lambda^2\mu^2 - \alpha_n^2) = 0$, with

$$\lambda \ll E^{\frac{1}{2}}, \quad S \ll (E/\lambda)^2,$$

which put us in sectors III and IV of figure 1. This is the Stokes layer in which, due to its extreme thinness, viscous stresses completely overwhelm Coriolis and buoyancy forces.

Let us now briefly review the core dynamics relevant to the various sectors of figure 1.

In sector I, the Stewartson layer dynamics is the only relevant dynamics and we have

$$\psi_n(r) \sim (q_n/6\lambda\gamma_n\alpha_n)\{F(\gamma_n r) - e^{-\frac{1}{3}i\pi}F(e^{\frac{1}{3}i\pi}\gamma_n r) - e^{\frac{1}{3}i\pi}F(e^{-\frac{1}{3}i\pi}\gamma_n r)\}, \tag{2.8}$$

where $\gamma_n = (2\alpha_n/E\lambda)^{\frac{1}{3}}$ and for convenience we introduced the function

$$F(x) = xK_1(x), \tag{2.9}$$

where the argument x can be complex. Meridional motion redistributes the basic angular momentum and temperature fields and thus drives Poisson equations for $A_n(r)$ and $H_n(r)$. The heat flux is composed of a boundary-layer contribution plus an interior contribution. Because $\psi \sim O(E/\lambda^2)^{\frac{1}{3}}$ (see table 1) from the heat equation we deduce that the boundary-layer contribution is

$$(S/E)\psi_n(r) \sim O(S/(E\lambda)^{\frac{2}{3}});$$

Hence to lowest order the heat flux and temperature distributions are just those given by conduction. The stratification is so weak that advection of the basic temperature field is negligible to lowest order so that

$$H_n(r) \sim -q_n F(\alpha_n r). \tag{2.10}$$

The angular momentum is also made up of a boundary-layer and an interior contribution and reads

$$A_n(r) \sim (q_n/2\alpha_n)\{F(\alpha_n r) - \tfrac{1}{3}[F(\gamma_n r) + F(\gamma_n r e^{\frac{1}{3}i\pi}) + F(\gamma_n r e^{-\frac{1}{3}i\pi})]\}. \tag{2.11}$$

Thus, whenever S, λ lie in sector I, the meridional motion consists of a closed gyre within a distance $(E\lambda)^{\frac{1}{3}}$ of the axis, the isotherms are determined conductively; the angular momentum distribution is $O(1)$ in the interior and is brought to zero in the Stewartson layer. Because both boundaries at $z = 0, 1$ are free surfaces, the $E^{\frac{1}{4}}$-Stewartson layer is absent; its existence relies on vortex tube stretching via Ekman-layer suction.

Sector II of figure 1 seems to be the region of most physical interest. Buoyancy and hydrostatic core layers when combined have three degrees of freedom and can account for all of the axial boundary conditions. The stream function is composed of a hydrostatic part and a buoyancy part, namely,

$$\psi_n(r) \sim (Eq_n/S)\{F(\mu_n r) - \tfrac{1}{2}F(yre^{\frac{1}{4}i\pi}) - \tfrac{1}{2}F(yre^{-\frac{1}{4}i\pi})\}, \tag{2.12}$$

where $\mu_n = 2\alpha_n/S^{\frac{1}{2}}$ and $y = S^{\frac{1}{4}}/(E\lambda)^{\frac{1}{2}}$ are the inverse length scales appropriate to the hydrostatic and buoyancy layers. Meridional motion consists of two parts; the heat sources near the axis induce vertical motions in the buoyancy layer. These vertical motions create lateral motions in the thicker hydrostatic layer. Note that there is no meridional motion outside the hydrostatic layer so that the radial extent of a meridional cell is $O(S^{\frac{1}{2}})$. Depending on the size of S, this may be smaller, comparable or larger than the radial conduction distance which is $O(1)$.

Next, solving for H_n we get

$$H_n(r) \sim -\tfrac{1}{2}q_n[F(yre^{\frac{1}{4}i\pi}) + F(yre^{-\frac{1}{4}i\pi})] + [4q_n/(4-S)]\,[F(\mu_n r) - F(\alpha_n r)]. \tag{2.13}$$

Thus, the heat flux consists of three parts. The final term is the heat flux that would be present if conduction were the only mechanism that could transfer heat. The remaining terms result from the advection of the basic temperature field in the buoyancy layer and the hydrostatic layer. Finally, the angular momentum reads

$$A_n(r) \sim [2q_n/\alpha_n(4-S)]\,[F(\alpha_n r) - F(\mu_n r)]. \tag{2.14}$$

To lowest order, angular momentum in the buoyancy layer is negligible and A_n consists of an interior contribution $F(\alpha_n r)$, which is the thermal wind associated with the conduction temperature profile, plus a hydrostatic contribution which is forced by the advection of the basic angular momentum by meridional motions.

Next, consider sector III of figure 1. The relevant layers are now the upwelling layer and the Stokes layer. Here

$$\psi_n(r) \sim (\tfrac{1}{4}iq_n)\,[F(\omega re^{\frac{1}{4}i\pi}) - F(\omega re^{-\frac{1}{4}i\pi})], \tag{2.15}$$

where $\omega = (2/E)^{\frac{1}{2}}$. Only the upwelling contribution is included since the contribution in the Stokes layer is $O(\lambda^2/E)$ which is negligible† in sector III. (Note that

† The neglected term is $(i\lambda^2/E)q_n F(\alpha_n r/\lambda)$.

such a layer is needed to bring $\partial w/\partial r$ to zero at the axis.) The vertical velocity is largest in the Stokes layer. Again, the heat flux and angular momentum distributions are

$$H_n(r) \sim q_n F(\alpha_n r) + O(S/E) \tag{2.16}$$

and

$$A_n(r) \sim (q_n/2\alpha_n)\{F(\alpha_n r) - \tfrac{1}{2}F(\omega r e^{\frac{1}{4}i\pi}) - \tfrac{1}{2}F(\omega r e^{-\frac{1}{4}i\pi})\}. \tag{2.17}$$

As in sector I, the stratification is so weak that conduction of heat occurs to lowest order. The angular momentum distribution is adjusted to zero within the upwelling layer.

Sector III is similar to sector I. The Stewartson layer of sector I splits into the upwelling layer and the Stokes layer. The role of the latter is simply to adjust the vertical velocity w to satisfy the boundary condition at $r = 0$.

Finally, region IV consists of three layers: the hydrostatic, the Stokes, and the viscous hydrostatic layers. Here

$$\psi_n(r) \sim (Eq_n/S)\{F(\mu_n r) - F(m_n r)\}, \tag{2.18}$$

where $\mu_n = 2\alpha_n/S^{\frac{1}{2}}$ and $m_n = S^{\frac{1}{2}}(\alpha_n E)$ are the decay rates associated with the hydrostatic and viscous hydrostatic layer respectively. The contribution due to the Stokes layer is again negligible† in so far as ψ_n is concerned. The heat flux and angular momentum distributions are found to be

$$H_n(r) \sim -[4q_n/(4-S)]\{F(\alpha_n r) - F(\mu_n r) + F(m_n r)\} \tag{2.19}$$

and

$$A_n(r) \sim [2q_n/(4-S)]\{F(\alpha_n r) - F(\mu_n r)\}. \tag{2.20}$$

Sector IV is then quite similar to sector III. The lowest-order angular momentum is identical; the Stokes and viscous hydrostatic layers take over the roles of the buoyancy layer.

Thus, from the above we see the wealth of possible axial structures that could coexist in a vortex when the surroundings are stably stratified. For $\beta \neq 0$ the lower boundary will exert a torque on the interior and therefore an Ekman layer will be present. Therefore $E^{\frac{1}{4}}$ layers, which rely on Ekman stretching, will also be present in the core. Aside from this added complication the core dynamics preserves the same character for $\beta \neq 0$.

3. Case 2, $\beta \neq 0$

Before solving the differential equations, let us anticipate some results pertaining to the angular momentum. The boundary condition at $z = 0$ tells us that angular momentum can be gained or lost through $z = 0$ via viscous torques. Whenever $A > 0$ (< 0), near $z = 0$, there is a loss (gain) of angular momentum. Because of the prescribed boundary condition at the free surface no angular momentum can be imparted or lost at $z = 1$.

Consider now an annular ring of fluid, extending from $z = 0$ to $z = 1$ and from r to $r + \Delta r$. Let $\bar{A}$ be the vertical average of $A(r, z)$. To be specific, suppose

$$A(r, 0) > 0$$

(an analogous discussion follows if $A(r, 0) < 0$). There is a viscous torque at $z = 0$ that is trying to decrease the value of $\bar{A}$. To maintain a steady flow we must have

† The neglected term is $(q_n \lambda^2/E\alpha_n^4)F(\alpha_n r/\lambda)$.

a net flux of angular momentum into the annular ring. Advection of angular momentum is ruled out by the linearization of the equations and the vanishing of the vertical velocity at $z = 0, 1$; hence, radial diffusion of angular momentum must negate the loss of angular momentum through $z = 0$. We can put this statement into a mathematical form by integrating (1.8b) from $z = 0$, to $z = 1$ and applying the boundary conditions; we get

$$D_1^2 \bar{A} = \beta A(r, 0). \tag{3.1}$$

Equation (3.1) has for a formal solution

$$\bar{A} = -\tfrac{1}{2}\beta \left\{ r^2 \int_r^\infty A(y, 0) \frac{dy}{y} + \int_s^r y A(y, 0)\, dy \right\}, \tag{3.2}$$

where the boundary conditions at $r = 0$ are satisfied and

$$(\bar{A})_{r=\infty} = -\tfrac{1}{2}\beta \int_0^\infty y A(y, 0)\, dy = 0, \tag{3.3}$$

since $\bar{A} \to 0$ as $r \to \infty$. We see that $A(r, 0)$ cannot have one sign for all values of r: the net torque acting at $z = 0$ must be zero for a steady state to prevail. After substituting (3.3) into (3.2), we have

$$\bar{A} = \tfrac{1}{2}\beta \int_r^\infty (y^2 - r^2) A(y, 0) \frac{dy}{y}. \tag{3.4}$$

Suppose we find $A(r, 0) > 0$ for $r < r_0$ and $A(r, 0) < 0$ for $r > r_0$. Then, for all $r \geqslant r_0$, we have $\bar{A} < 0$. Thus the region $r > r_0$ is the 'surroundings' of the vortex cell having a horizontal dimension of r_0.

Now, we shall investigate the structure of the solution when $\beta \neq 0$. The change in boundary conditions at $z = 0$ complicates the problem since a simple representation for the solution (e.g. (2.1)) cannot be found. Consequently, z derivatives (as well as r derivatives) will be involved in boundary-layer analysis near $r = 0$. Moreover an Ekman layer exists at $z = 0$. By restricting our attention to that part of the fluid outside the Ekman layer, we may use the compatibility conditions† at $z = 0$ whenever $S \ll E^{-\frac{1}{2}}$ and the radial scales involved are larger than $E^{\frac{1}{2}}$. The general compatibility condition at $z = 0$ can be found to be

$$\frac{\partial \psi}{\partial r} = \frac{-E}{2(1+\beta^2 E)} \left(\frac{\partial}{\partial z} - \beta \right) \frac{\partial A}{\partial r}. \tag{3.5}$$

This condition when applied to the hydrostatic interior simplifies to

$$\psi(r, 0) = \frac{\beta E}{2(1+\beta^2 E)} A(r, 0), \quad \theta(r, 0) = 0,$$

and if $0 \leqslant \beta \ll E^{-\frac{1}{2}}$ the above reads

$$\psi(r, 0) = \tfrac{1}{2} E \beta A(r, 0), \quad \theta(r, 0) = 0. \tag{3.6}$$

There is no Ekman layer at $z = 1$.

† See, for example, Greenspan (1968, p. 46).

For reasons to be apparent shortly, we restrict our attention to values of the stratification parameter satisfying $E^{\frac{1}{2}} \ll S \ll E^{-\frac{1}{2}}$. Except for the Ekman layer, the fluid consists of two regions, a non-hydrostatic core region (the buoyancy-layer region) and a hydrostatic region. We call the 'interior' here any hydrostatic region, regardless of its radial scale. The purpose is to determine the flow and heat flux distributions of the interior region. We write ψ, A, H as the sum of two parts: a non-hydrostatic part, denoted by a subscript 'N', and a hydrostatic part, denoted by a subscript 'H'. The boundary conditions at $r = 0$ are:

$$\begin{aligned} \psi_N+\psi_H &= A_N+A_H = 0, \\ H_N+H_H &= -q(z). \end{aligned} \tag{3.7}$$

Since the radial scale of the non-hydrostatic layer is much smaller than unity, we have, from the heat equation,

$$\psi_N = (E/S)H_N. \tag{3.8}$$

Since $S \gg E^{\frac{1}{2}}$, the thermal wind balance prevails in the hydrostatic region, i.e.

$$2(\partial A_H/\partial z) = H_H. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we find

$$\psi_H = (E/S)\,[q(z)-2\,\partial A_N/\partial z].$$

In case 1, we saw that in the buoyancy region the angular momentum distribution was negligible; therefore the boundary conditions on the hydrostatic region are

$$\psi_H = (E/S)q(z), \quad A_H = 0, \quad H_H = 0 \quad \text{at} \quad r = 0. \tag{3.10}$$

After dropping the subscript H we write the governing equations for the hydrostatic region as

$$\left.\begin{aligned} D^2\psi &= (E/S)D^2H, \\ 2\,\partial A/\partial z &= H, \\ -2\,\partial\psi/\partial z &= ED^2A, \end{aligned}\right\} \tag{3.11}$$

subject to (3.10) at $r = 0$ and

$$\left.\begin{aligned} &\psi = 0, && \partial A/\partial z = 0, \quad H = 0 \quad \text{at} \quad z = 1, \\ &\psi = \tfrac{1}{2}E\beta A, && \partial A/\partial z = 0, \quad H = 0 \quad \text{at} \quad z = 0. \end{aligned}\right\} \tag{3.12}$$

The vanishing of $\partial A/\partial z$ at $z = 0$ derives from the absence of a boundary layer for θ and therefore $\theta = H = 0$ at the edge of the Ekman layer. A single separation of variable does not lead to a solution. However, we can divide the hydrostatic problem into two parts

$$(\psi, A, H) = (\psi_0, A_0, H_0)+(\psi_1, A_1, H_1), \tag{3.13}$$

where the first term is just the solution we would have if $\beta = 0$, i.e.

$$\left.\begin{aligned} \psi_0(r,z) &= (E/S)\sum_{n=1}^{\infty} q_n F(2n\pi r/S^{\frac{1}{2}})\sin n\pi z, \\ A_0(r,z) &= (2/(4-S))\sum_{n=1}^{\infty} \frac{q_n}{n\pi}\,[F(n\pi r)-F(2n\pi r/S^{\frac{1}{2}})]\cos n\pi z, \\ H_0(r,z) &= -(4/(4-S))\sum_{n=1}^{\infty} q_n[F(n\pi r)-F(2n\pi r/S^{\frac{1}{2}})]\sin n\pi z. \end{aligned}\right\} \tag{3.14}$$

The solutions (ψ_1, A_1, H_1) satisfy the same differential equations (3.11); the boundary conditions are

$$\left.\begin{aligned} &\psi_1 = 0, \quad H_1 = 0, \quad A_1 = 0 \quad \text{at} \quad r = 0,\\ &\psi_1 = 0, \quad H_1 = 0, \quad \partial A_1/\partial z = 0 \quad \text{at} \quad z = 1,\\ &\psi_1 - \tfrac{1}{2}E\beta A_1 = \tfrac{1}{2}E\beta A_0, \quad H_1 = \partial A/\partial z = 0 \quad \text{at} \quad z = 0 \end{aligned}\right\} \tag{3.15}$$

For the $(\ )_1$ problem, the forcing is now at $z = 0$. Take a Hankel transform in the r direction, and solve the resulting ordinary differential equation in z with the appropriate boundary conditions at $z = 0, 1$. The result is an integral representation for ψ_1, A_1, H_1, viz.

$$\left.\begin{aligned} A_1(r,z) &= \int_0^\infty (kr)J_1(kr)\hat{A}_0(k,0)\,[b_1(k)\cosh k(1-z) + b_2(k)\cosh \gamma k(1-z)]\,dk,\\ \psi_1(r,z) &= \frac{\gamma^2-1}{2\gamma}\int_0^\infty (kr)J_1(kr)\hat{A}_0(k_1 0)\,[kb_2(k)\sinh \gamma k(1-z)]\,dk,\\ H_1(r_1 z) &= -2\int_0^\infty (kr)J_1(kr)\hat{A}_0(k_1 0)\,[kb_1(k)\sinh k(1-z) + \gamma k b_2(k)\sinh \gamma k(1-z)]\,dk, \end{aligned}\right\} \tag{3.16}$$

where

$$\left.\begin{aligned} &\gamma = (\tfrac{1}{4}S)^{\frac{1}{2}}, \quad \hat{A}_0(k,0) = \int_0^\infty A_0(r,0)J_1(kr)\,dr,\\ &b_1(k) = -\beta\gamma^2 \sinh \gamma k\{(\gamma^2-1)k\sinh \mu k - \beta\gamma\cosh \gamma k \sinh k + \beta\gamma^2 \sinh \gamma k \cosh k\}^{-1},\\ &b_2(k) = -[b_1(k)\sinh k]/[\gamma \sinh \gamma k]. \end{aligned}\right\} \tag{3.17}$$

Using (3.14), the definition for $F(x)$, and invoking Abramowitz & Stegun (1964) we can write $\hat{A}_0(k,0)$ as

$$\hat{A}_0(k,0) = k\sum_{n=1}^{\infty}\left(\frac{q_n}{2n\pi}\right)\frac{(n\pi)^2}{[(n\pi)^2+(k\gamma)^2]\,[(n\pi)^2+k^2]}. \tag{3.18}$$

For 'reasonable' heating functions $q(z)$ we anticipate that for $n \gg 1$, $q_n \sim n^{-2}$ and therefore $\hat{A}_0 \sim 1/n^5$ for a given k. We can then argue that the fundamental of $q(z)$ will be the dominant component. To illustrate we consider a simple example in which the axial heating is taken to be equal to $\sin \pi z$, i.e. we let

$$q_n \equiv \delta_{1_n} \quad (n = 1, 2, 3, \ldots) \tag{3.19}$$

where δ_{1_n} is the Kronecker delta.

After substituting (3.18), (3.19) into (3.16), we approximate these integrals by performing the numerical integration up to a sufficiently large value of k. Using (3.13) and (3.14) we find ψ, A, H. Lines of constant ψ, A, H are shown in figure 2 for $S = 8$ and $\beta = 0$, 1, 100.

For $\beta = 0$, we have two free surfaces and the flow is symmetrical about $z = 0{\cdot}5$. The non-hydrostatic layer (the buoyancy layer) screens the interior from the prescribed axial heat sources. These are felt in the interior in the form of an axial distribution of mass sinks and sources. From (3.8), the radial velocity at the edge of the buoyancy layer is proportional to $-\cos \pi z$ implying inward radial motion for $0 \leqslant z \leqslant 0{\cdot}5$ and outward radial motion for $0{\cdot}5 \leqslant z \leqslant 1{\cdot}0$. Then, the streamlines in the interior emanate from the core, for $0{\cdot}5 \leqslant z \leqslant 1{\cdot}0$, and return to the core, for $0 \leqslant z \leqslant 0{\cdot}5$, never touching the $z = 0, 1$ surfaces. This meridional recirculation redistributes the basic angular momentum, creating a region of negative relative angular momentum ($z > 0{\cdot}5$) and a region of positive relative angular momentum ($z < 0{\cdot}5$). Then, the line $z = 0{\cdot}5$ is the line of zero angular momentum. The meridional circulation also redistributes the basic temperature stratification and the lines H = const. are also symmetric about $z = 0{\cdot}5$.

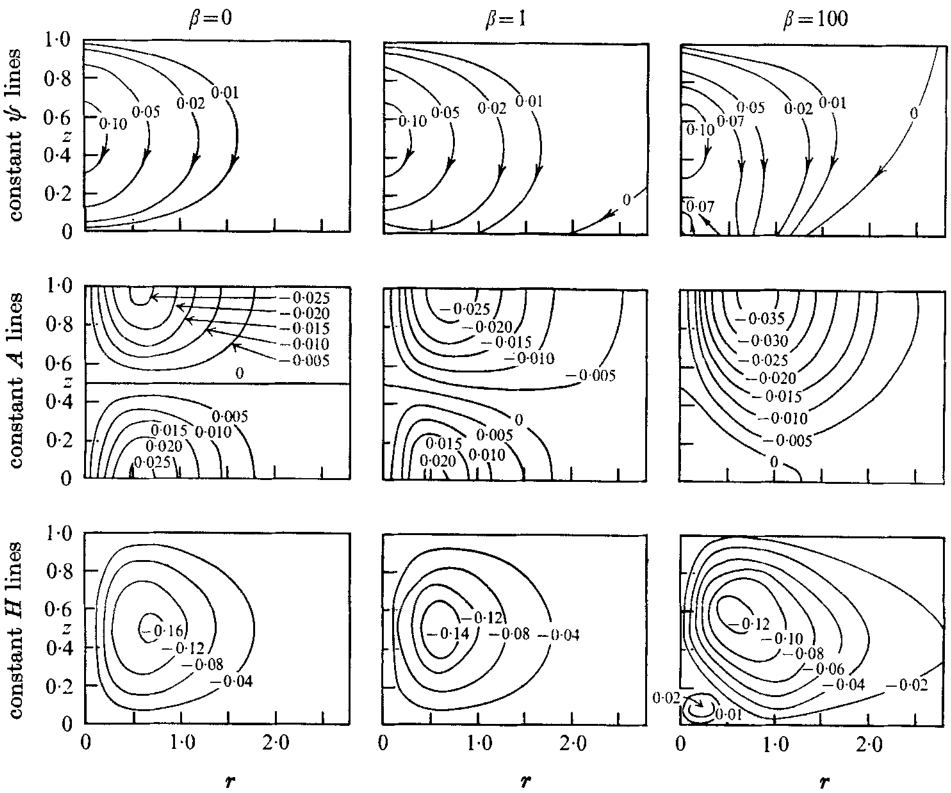

FIGURE 2. Lines of constant ψ, A, H for $S = 8$ and $\beta = 0, 1, 100$.

For $\beta = 1$, the $z = 0, 1$ boundaries no longer play the same role: angular momentum is lost at the lower boundary. To replenish this loss of angular momentum (which is only slight here) some of the interior stream tubes skim and dip into the lower Ekman layer. Now the $A = 0$ line bends downward and intersects the $z = 0$ surface at $r_0 = 1{\cdot}8$, enclosing a region for which $A > 0$. Finally, the heat flux lines H = const. are no longer symmetric since the symmetry in the streamline pattern is destroyed by viscous effects found at $z = 0$.

For $\beta = 100$, for all intents and purposes the lower boundary acts like a rigid surface. Now, the stream tubes penetrate deeply into the lower Ekman layer in which basic angular momentum is lost by viscous torques. $A(r, 0)$ increases, as we move away from $r = 0$, reaches a maximum, vanishes at $r = r_0$ and becomes negative for $r > r_0$. Therefore, positive vertical velocities are found at the edge of the Ekman layer in the vicinity of the axis. These velocities and the radial velocities found next to the axis are responsible for the peculiar behaviour found in the corner. In that region the angular momentum is small and positive with little or no r and z variations. The $A = 0$ line cuts the $z = 0$ surface at $r_0 = 1{\cdot}3$, i.e. the radial extent of that region is smaller than the one found in the previous case. The strong asymmetries in the streamline pattern induce asymmetric redistributions of the basic temperature field as shown by the lines $H = \text{const}$. The behaviour near the corner is a consequence of the streamline recirculation in that region.

4. Summary

The (λ, S) plane contains sectors in which several axial layers coexist and when combined yield an axial boundary-layer structure which satisfies three of the axial boundary conditions, the fourth one being satisfied by the diffuse interior. The extent to which the meridional cell penetrates in the interior depends upon the value of the parameters λ, S. By using torque considerations we were able to explain the presence and relative sizes of the cells of positive and negative relative angular momentum. Finally, very different flow behaviour is obtained for various values of β. For large β, the streamlines in the interior penetrate the Ekman layer in the far field and erupt out of that layer before reaching the axis.

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